Simple vs simple

Def: A test φ is most <u>powerful size α</u> if

a) φ is size α

b) for every other size α test φ^*, $E_\theta \varphi \geq E_\theta \varphi^*$

$\forall \theta \in \Theta - \omega$

The big testing theorem which is the basis for all testing is

Thm: (Neyman-Pearson lemma) Let X be an r.v. with unknown distribution either $f_0(x)$ or $f_1(x)$ and consider the test situation

$H_0: X \sim f_0(x)$ vs $H_1: X \sim f_1(x)$

(simple vs simple)

To test H_0 vs H_1 the most powerful size α test is

$$\varphi(x) = \begin{cases} 1 & x \ni f_0(x)/f_1(x) < k_\alpha \\ \gamma_\alpha & \qquad\qquad\qquad = k_\alpha \\ 0 & \qquad\qquad\qquad > k_\alpha \end{cases}$$

where k_α and γ_α are chosen such that $E_{f_0}\varphi(x) = \alpha$

Ex: $X_1, \ldots, X_5 \overset{iid}{\sim} B(1,p)$

$H_0: p = .5 \qquad H_1: p = .75$

$f_0(x)/f_1(x) = [1^{\Sigma x} 1^{5-\Sigma x}/2^5] / [3^{\Sigma x} 1^{5-\Sigma x}/4^5] = 2^5/3^{\Sigma x_i}$

To make $\alpha = .05$

$f_0(x)/f_1(x) = 2^5/3^{\Sigma x_i} \searrow$ in Σx_i

rej if Σx_i too big

Neyman-Pearson Lemma

Lemma 1.1 *Let X be a random variable with unknown distribution, either $f_0(x)$ or $f_1(x)$, and consider the test situation*

$$H_0 : X \sim f_0(x) \text{ versus } H_1 : X \sim f_1(x) \tag{1}$$

(simple versus simple). To test H_0 versus H_1, the most powerful size α test is

$$\varphi(x) = \begin{cases} 1 & x \ni f_0(x)/f_1(x) < k_\alpha \\ \gamma_\alpha & \qquad\qquad = k_\alpha \\ 0 & \qquad\qquad > k_\alpha \end{cases} \tag{2}$$

where k_α and γ_α are chosen such that $\mathrm{E}_{f_0}(\varphi(x)) = \alpha$.

This lemma is the basis for future testing where we move on to composite hypotheses.

Proof 1.1 *Consider two tests, φ and φ^*. The difference of the powers of these tests is*

$$\begin{aligned}
(1-\beta_\varphi) - (1-\beta_{\varphi^*}) &= \mathrm{E}_{f_1}(\varphi) - \mathrm{E}_{f_1}(\varphi^*) && (3)\\
&= \mathrm{E}_{f_1}(\varphi - \varphi^*) && (4)\\
&= \int_{\mathfrak{X}} (\varphi(x) - \varphi^*(x))\, f_1(x)dx && (5)\\
&= \int_{x \ni f_0/f_1 < k} (\varphi(x) - \varphi^*(x))\, f_1(x)dx + \int_{x \ni f_0/f_1 = k} (\varphi(x) - \varphi^*(x))\, f_1(x)dx \\
&\quad + \int_{x \ni f_0/f_1 > k} (\varphi(x) - \varphi^*(x))\, f_1(x)dx && (6)\\
&\geq \frac{1}{k}\int_{x \ni f_0/f_1 < k} (\varphi(x) - \varphi^*(x))\, f_0(x)dx + \frac{1}{k}\int_{x \ni f_0/f_1 = k} (\varphi(x) - \varphi^*(x))\, f_0(x)dx \\
&\quad + \frac{1}{k}\int_{x \ni f_0/f_1 > k} (\varphi(x) - \varphi^*(x))\, f_0(x)dx && (7)\\
&\geq \frac{1}{k_\alpha}\mathrm{E}_{f_0}(\varphi(X) - \varphi^*(X)) && (8)\\
&\geq 0 && (9)
\end{aligned}$$

Thus, the power of φ is no less than the power of φ^*.

The transition between alternative and null hypotheses in equations 6 and 7 is due to the fact that in the first integral we are in the rejection region so $\varphi - \varphi^* = 1 - \varphi^* \geq 0$ and $f_0/f_1 < k$ means that $f_1 \geq f_0/k$. Similarly, in the middle integral we are in the randomization region so $f_0/f_1 = k$ means that $f_1 = f_0/k$. Finally, for the last integral we are in the "acceptance" region so $\varphi - \varphi^* = 0 - \varphi^* \leq 0$ and $f_0/f_1 > k$ means that $f_1 \geq f_0/k$.

Equation 9 follows from 8 follows since $k_\alpha \geq 0$ and under H_0 we have that $\mathrm{E}_0(\varphi(X)) \geq \mathrm{E}_0(\varphi^*(X))$ or $\alpha \geq \alpha^*$ by size.

Example 1.1 *Let $X_i \overset{iid}{\sim} B(1,p)$. To test*

$$H_0 : p = 0.5 \text{ versus } H_1 : p = 0.75$$

we look at

$$\frac{f_0(x)}{f_1(x)} = \frac{1^{\sum x_i} 1^{5-\sum x_i}/2^5}{3^{\sum x_i} 1^{5-\sum x_i}/4^5} \tag{10}$$

$$= \frac{2^5}{3^{\sum x_i}} \tag{11}$$

To make $\alpha = 0.05$ we note that

$$\frac{f_0(x)}{f_1(x)} = \frac{2^5}{3^{\sum x_i}} \tag{12}$$

is decreasing for increasing $t = \sum x_i$. So we will reject H_0 if $T = \sum X_i$ is too large. Now note that $T = \sum X_i \sim B(5,p)$ so that under the null hypothesis

$$f_0(t) = \binom{5}{t}\left(\frac{1}{2}\right)^t \left(\frac{1}{2}\right)^{5-t} \tag{13}$$

Thus

$t = \sum x_i$	0	1	2	3	4	5
$f_0(t)$	0.0313	0.1563	0.3125	0.3125	0.1563	0.0313

If we reject for $\sum x_i = 5$ ($f_0/f_1 = 0.1317$) then we reject with probability 0.0313 under H_0 (not enough). Using $\sum x_i \geq 4$ ($f_0/f_1 = .3951$) we reject with probability $0.1563 + 0.0313 = 0.1876$ (too much). So, we should reject if $\sum x_i = 5$ and randomize for $\sum x_i = 4$. To get exact size α we need γ_α.

Note that

$$\mathrm{E}_{1/2}(\varphi(X)) = \mathrm{P}_{1/2}[\text{reject } H_0] \tag{14}$$

$$= \gamma \mathrm{P}_{1/2}\left(\sum X_i = 4\right) + \mathrm{P}_{1/2}\left(\sum X_i = 5\right) \tag{15}$$

$$= \gamma 0.1563 + 0.0313 \tag{16}$$

$$= 0.05 \tag{17}$$

Solving for γ we obtain $\gamma = 0.1196$ and our level $\alpha = 0.05$ test is

$$\varphi(\mathbf{X}) = \begin{cases} 1 & \sum X_i = 5 \\ 0.1196 & \sum X_i = 4 \\ 0 & \text{else} \end{cases} \tag{18}$$

Finally, note that if $p = \frac{3}{4}$, the power of this test is

$$1 - \beta = \left(\frac{3}{4}\right)^5 + \gamma \binom{5}{4}\left(\frac{3}{4}\right)^4 \left(\frac{1}{4}\right) \tag{19}$$

$$= 0.2373 + \gamma 0.3955 \tag{20}$$

$$= 0.2846 \tag{21}$$

which is pretty low. We might want to use a few more observations.

Do this

Power φ Power φ^*

pf: $(1-\beta_\varphi)-(1-\beta_{\varphi^*}) = E_{f_1}(\varphi) - E_{f_1}(\varphi^*)$

$= E_{f_1}(\varphi - \varphi^*)$

$= \int_* (\varphi(x) - \varphi^*(x)) f_1(x)\,dx$

$$= \underbrace{\int_{* \ni f_0/f_1 < k} (\overset{1 \leftarrow rej}{\varphi} - \varphi^*) f_1(x)\,dx}_{\text{rej reg } \geq 0} + \int_{* \ni f_0/f_1 = k} (\varphi - \varphi^*) f_1(x)\,dx + \underbrace{\int_{* \ni f_0/f_1 > k} (\overset{0 \leftarrow acc}{\varphi} - \varphi^*) f_1(x)\,dx}_{\text{"acc" reg } \leq 0}$$

$$\geq \frac{1}{k} \int_{* \ni f_0/f_1 < k} (\varphi - \varphi^*) f_0(x)\,dx + \frac{1}{k} \int_{* \ni f_0/f_1 = k} (\varphi - \varphi^*) f_0(x)\,dx + \frac{1}{k} \int_{* \ni f_0/f_1 > k} (\varphi - \varphi^*) f_0(x)\,dx$$

$= \frac{1}{k_\alpha} E_{f_0}(\varphi(x) - \varphi^*(x))$

≥ 0 since $k_\alpha \geq 0$ and $E_0(\varphi(x)) \geq E_0(\varphi^*(x))$ or $\alpha \geq \alpha^*$ by size

Thus power of $\varphi \geq$ power φ^*

Ex: Let $X_1, \ldots, X_{10} \overset{iid}{\sim} \mathcal{E}(\lambda)$

$H_0: \lambda = \frac{1}{2}$ $H_1: \lambda = 2$

$f_0/f_1 = \left[\frac{1}{2^{10}} e^{-\Sigma x_i/2}\right] / \left[2^{10} e^{-\Sigma x_i 2}\right]$

$= e^{\frac{3}{2}\Sigma x_i} / 4^{10}$

rej H_0 if ΣX_i small $\iff f_0/f_1$ small

Under H_0 $X_i \overset{iid}{\sim} \mathcal{E}(\frac{1}{2}) \overset{joint}{\Rightarrow}$ is $\chi^2(20)$ $\Gamma(\frac{1}{2}, \frac{N}{2}) = \chi^2_N$

χ^2_{20} table says $\alpha = .05$ if $\Sigma X_i = 10.85$

so rej H_0 if $\Sigma X_i \leq 10.85$

or rej H_0 if $\bar{X} < 1.085$

$$\varphi = \begin{cases} 1 & \bar{X} < 1.085 \\ 0 & \bar{X} \geq 1.085 \end{cases}$$

Exp'l dist = Gamma

Power? $1-\beta = P(\text{rej } H_0 \mid H_1) = \int_0^{10.85} (t^9 2^{10} e^{-2t} / \Gamma(10))\,dt$ (numerically)

or $\bar{X} \sim N(\frac{1}{\lambda}, \frac{1}{n\lambda^2}) \Rightarrow$ under H_1 $\bar{X} \sim N(\frac{1}{2}, \frac{1}{40})$

$1-\beta = P(\bar{X} < 1.085 \mid H_1) = P\left(z < \frac{1.085 - .5}{\sqrt{1/40}}\right) = P(z \leq 3.7) \approx .9999$